

Week 8 at a glance

We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
 - Defining important sets of numbers, e.g. set of integers, set of rational numbers
 - Defining functions using multiple representations
 - Classifying sets into: finite sets, countably infinite sets, uncountable sets
 - Using functions to compare cardinality of sets
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems. Use mathematical techniques to solve problems. Determine appropriate conceptual tools to apply to new situations. Know when tools do not apply and try different approaches. Critically analyze and evaluate candidate solutions.
 - Determining what evidence is required to establish that a quantified statement is true or false
 - Evaluating quantified statements about finite and infinite domains
- Apply proof strategies, including direct proofs and proofs by contradiction, and determine whether a proposed argument is valid or not.
 - Tracing and/or modifying a proof by contradiction
 - Using proofs as knowledge discovery tools to decide whether a statement is true or false

TODO:

Review quiz based on Week 7 class material (due Monday 02/23/2026)

Homework 4 due on Gradescope <https://www.gradescope.com/> (Thursday 02/26/2026)

Review quiz based on Week 8 class material (due Monday 03/02/2026)

Test 2 Attempt 1 in the CBTF next week at your scheduled time.

Week 8 Monday: Cardinality of Sets

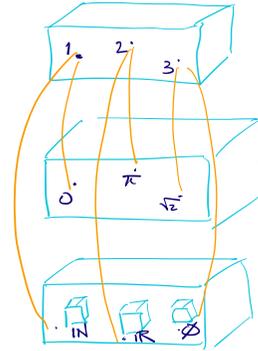
Definition: A **finite** set is one whose distinct elements can be counted by a **natural number**.

Motivating question: when can we say one set is *bigger than* another?

integer greater than or equal to zero

Which is bigger?

- The set $\{1, 2, 3\}$ or the set $\{0, 1, 2, 3\}$? *Bigger*
- The set $\{0, \pi, \sqrt{2}\}$ or the set $\{\mathbb{N}, \mathbb{R}, \emptyset\}$? *Neither!*
- The set \mathbb{N} or the set \mathbb{R}^+ ? *??*

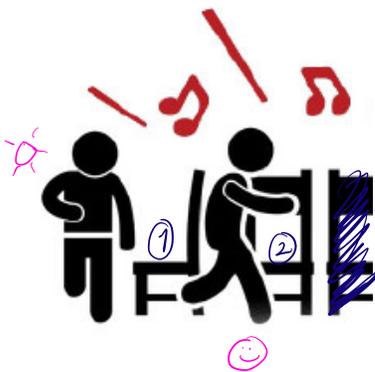


ex.
 $\{ \{1, 2, 3\} \}$
 has one element

Which of the sets above are **finite?** **infinite?**

Key idea for cardinality: Counting distinct elements is a way of labelling elements with natural numbers. **This is a function!** In general, functions let us associate elements of one set with those of another. If the association is “good”, we get a correspondence between the elements of the subsets which can relate the sizes of the sets.

Analogy: Musical chairs



People try to sit down when the music stops

Person☼ sits in Chair 1, Person☹ sits in Chair 2,

Person☹ is left standing!

What does this say about the number of chairs and the number of people?

Fewer chairs than people.

More people than chairs

Recall that a function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain. The domain and codomain are nonempty sets. The rule can be depicted as a table, formula, English description, etc.

A function can *fail to be well-defined* if there is some domain element where the function rule doesn't give a unique codomain element. For example, the function rule might lead to more than one potential image, or to an image outside of the codomain.

Example: $f_A : \mathbb{R}^+ \rightarrow \mathbb{Q}$ with $f_A(x) = x$ is **not** a well-defined function because

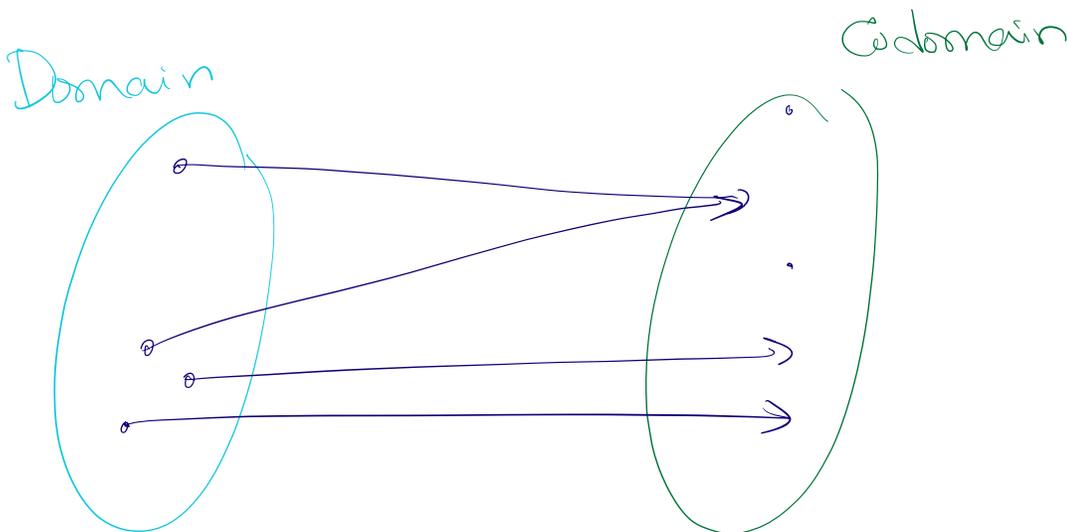
Counterexample $x = \sqrt{2}$ is in the domain \mathbb{R}^+
 but $f_A(\sqrt{2}) = \sqrt{2}$ is not in the codomain \mathbb{Q}
 (by Friday's argument).

Example: $f_B : \mathbb{Q} \rightarrow \mathbb{Z}$ with $f_B\left(\frac{p}{q}\right) = p + q$ is **not** a well-defined function because

Counterexample $x = \frac{1}{2} = \frac{4}{8}$ is in domain \mathbb{Q} $\frac{1}{2} = \frac{4}{8}$
 $f_B\left(\frac{1}{2}\right) = 1 + 2 = 3$
 $f_B\left(\frac{4}{8}\right) = 4 + 8 = 12$ \leftarrow Not the same!

Example: $f_C : \mathbb{Z} \rightarrow \mathbb{R}$ with $f_C(x) = \frac{x}{|x|}$ is **not** a well-defined function because

Counterexample $x = 0$ is in the domain \mathbb{Z}
 but $f_C(x) = \frac{x}{|x|} = \frac{0}{|0|} = \frac{0}{0}$ is not defined!



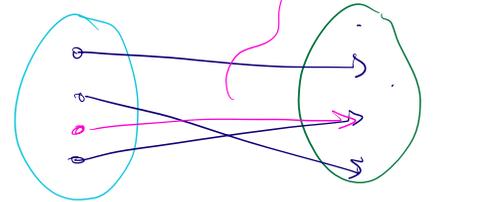
Definition : A function $f : D \rightarrow C$ is **one-to-one** (or **injective**) means for every a, b in the domain D , if $f(a) = f(b)$ then $a = b$.

Formally, $f : D \rightarrow C$ is one-to-one means $\forall a \in D \forall b \in D (f(a) = f(b) \rightarrow a = b)$

Informally, a function being one-to-one means “no duplicate images”.

Contrapositive

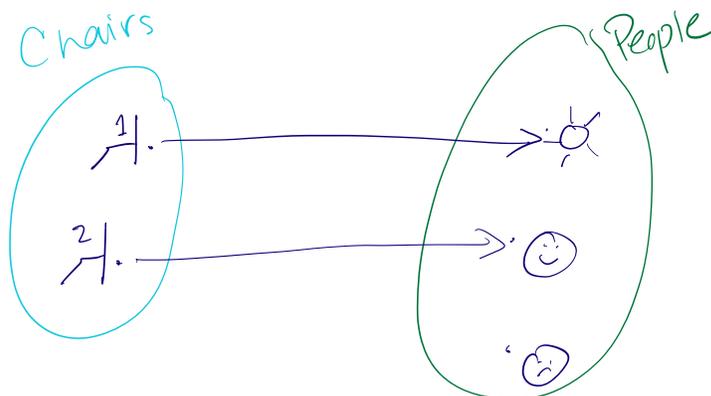
$$\forall a \in D \forall b \in D (a \neq b \rightarrow f(a) \neq f(b))$$



Definition: For nonempty sets A, B , we say that **the cardinality of A is no bigger than the cardinality of B** , and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B . Also, we define $|\emptyset| \leq |B|$ for all sets B .

In the analogy: The function $sitter : \{Chair1, Chair2\} \rightarrow \{Person\star, Person\ominus, Person\odot\}$ given by $sitter(Chair1) = Person\star$, $sitter(Chair2) = Person\ominus$, is one-to-one and witnesses that

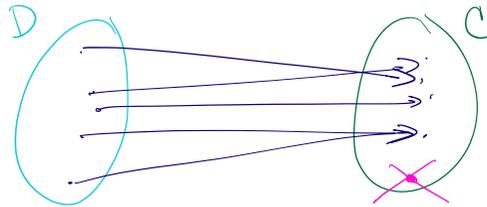
$$|\{Chair1, Chair2\}| \leq |\{Person\star, Person\ominus, Person\odot\}|$$



Definition: A function $f : D \rightarrow C$ is **onto** (or **surjective**) means for every b in the codomain, there is an element a in the domain with $f(a) = b$.

Formally, $f : D \rightarrow C$ is onto means $\forall b \in C \exists a \in D (f(a) = b)$

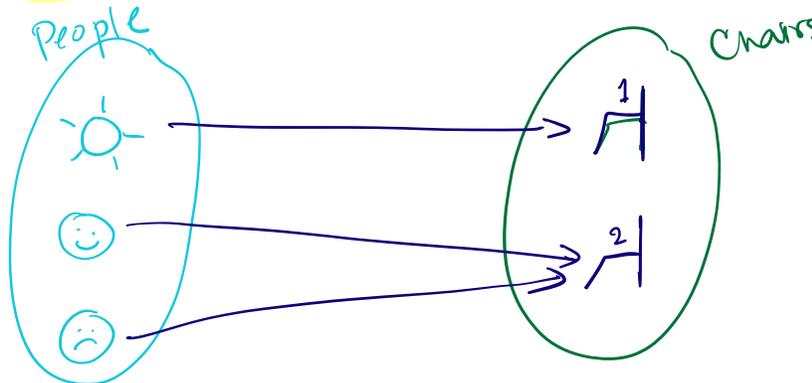
Informally, a function being onto means “every potential image is an actual image”.



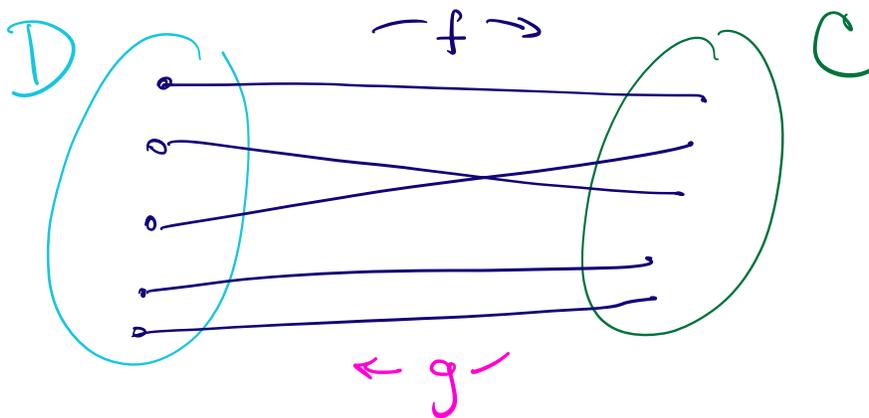
Definition: For nonempty sets A, B , we say that **the cardinality of A is no smaller than the cardinality of B** , and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B . Also, we define $|A| \geq |\emptyset|$ for all sets A .

In the analogy: The function $triedToSit : \{Person^{\odot}, Person^{\ominus}, Person^{\otimes}\} \rightarrow \{Chair1, Chair2\}$ given by $triedToSit(Person^{\otimes}) = Chair1$, $triedToSit(Person^{\ominus}) = Chair2$, $triedToSit(Person^{\odot}) = Chair2$, is onto and witnesses that

$$|\{Person^{\otimes}, Person^{\ominus}, Person^{\odot}\}| \geq |\{Chair1, Chair2\}|$$



Definition : A function $f : D \rightarrow C$ is a **bijection** means that it is both **one-to-one** and **onto**. The **inverse** of a bijection $f : D \rightarrow C$ is the function $g : C \rightarrow D$ such that $g(b) = a$ iff $f(a) = b$.

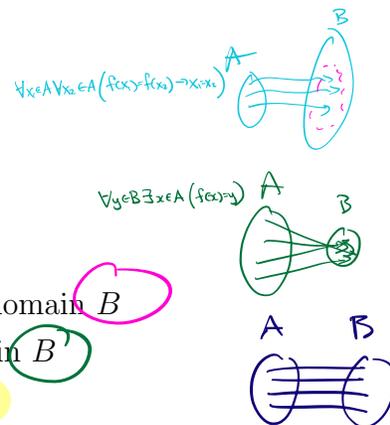


Week 8 Wednesday and Friday: Finite, countably infinite, and uncountable sets

Cardinality of sets

For nonempty sets A, B we say

- $|A| \leq |B|$ means there is a one-to-one function with domain A , codomain B
- $|A| \geq |B|$ means there is an onto function with domain A , codomain B
- $|A| = |B|$ means there is a bijection with domain A , codomain B



For all sets A , we say $|A| = |\emptyset|$, $|\emptyset| = |A|$ if and only if $A = \emptyset$.

Caution: we use familiar symbols to define cardinality, like $| \leq |$ and $| \geq |$ and $| = |$, but the meaning of these symbols depends on context. We've seen that vertical lines can mean absolute value (for real numbers), divisibility (for integers), and now sizes (for sets).

Now we see that \leq and \geq can mean comparing numbers or comparing sizes of sets. When the sets being compared are finite, the definitions of $|A| \leq |B|$ agree.

But, properties of numbers cannot be assumed when comparing cardinalities of infinite sets.

In a nutshell: cardinality of sets is defined via functions. This definition agrees with the usual notion of "size" for finite sets.

Properties of cardinality

$\forall A (|A| = |A|)$ *Every set is the same size as itself*
 $\forall A \forall B (|A| = |B| \rightarrow |B| = |A|)$
 $\forall A \forall B \forall C ((|A| = |B| \wedge |B| = |C|) \rightarrow |A| = |C|)$

$f_1: A \rightarrow B$ bijection $f_2: B \rightarrow C$ bijection *WTS $\exists g: A \rightarrow C$ (g is bijection)*
Strategy: Define $g(a) = f_2(f_1(a))$

Extra practice with proofs: Use the definitions of bijections to prove these properties.

Cantor-Schroder-Bernstein Theorem: For all nonempty sets,

$$|A| = |B| \quad \text{if and only if} \quad (|A| \leq |B| \text{ and } |B| \leq |A|) \quad \text{if and only if} \quad (|A| \geq |B| \text{ and } |B| \geq |A|)$$

find bijection
find two functions, each of which is injection
find two functions, each of which is surjection

To prove $|A| = |B|$, we can do any **one** of the following

- Prove there exists a bijection $f : A \rightarrow B$;
- Prove there exists a bijection $f : B \rightarrow A$;
- Prove there exists two functions $f_1 : A \rightarrow B$, $f_2 : B \rightarrow A$ where each of f_1, f_2 is one-to-one.
- Prove there exists two functions $f_1 : A \rightarrow B$, $f_2 : B \rightarrow A$ where each of f_1, f_2 is onto.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\} \\ = \{x \in \mathbb{Z} \mid x \geq 0\}$$

Definition: A set A is countably infinite means it is the same size as \mathbb{N} .

Natural numbers \mathbb{N}

List: 0 1 2 3 4 5 6 7 8 9 10...

identity : $\mathbb{N} \rightarrow \mathbb{N}$ with identity(n) = n

Claim: identity is a bijection. Proof: Ex.

well defined
one to one
onto

Corollary: $|\mathbb{N}| = |\mathbb{N}|$

Positive integers \mathbb{Z}^+

List: 1 2 3 4 5 6 7 8 9 10 11...

positives : $\mathbb{N} \rightarrow \mathbb{Z}^+$ with positives(n) = $n + 1$

Claim: positives is a bijection. Proof: Ex.

well defined
one to one
onto

Corollary: $|\mathbb{N}| = |\mathbb{Z}^+|$

Negative integers \mathbb{Z}^-

List: -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11...

negatives : $\mathbb{N} \rightarrow \mathbb{Z}^-$ with negatives(n) = $-n - 1$

Claim: negatives is a bijection.

Corollary: $|\mathbb{N}| = |\mathbb{Z}^-|$

Proof: We need to show it is a well-defined function that is one-to-one and onto.

• Well-defined?

Consider an arbitrary element of the domain, $n \in \mathbb{N}$. We need to show it maps to exactly one element of \mathbb{Z}^- .

By definition of \mathbb{N} , $n \geq 0$.
negatives(n) = $-n - 1$ so WTS $-n - 1 \in \mathbb{Z}^-$.
Starting with $n \geq 0$, multiplying both sides by -1 flips inequality: $-n \leq 0$
And then subtracting 1 from both sides gives $-n - 1 \leq -1 < 0$, as required. \square

• One-to-one?

Consider arbitrary elements of the domain $a, b \in \mathbb{N}$. We need to show that

$$(negatives(a) = negatives(b)) \rightarrow (a = b)$$

Assume, towards direct proof, that negatives(a) = negatives(b).
By definition this means $-a - 1 = -b - 1$. Adding 1 to both sides and then multiplying both sides by -1 gives $a = b$. \square

• Onto?

Consider arbitrary element of the codomain $b \in \mathbb{Z}^-$. We need witness in \mathbb{N} that maps to b .

Let $a = -b - 1$. Since $b < 0$, $a \geq 1$ so it is in \mathbb{N} .
Applying the function definition, negatives(a) = $-(-b - 1) - 1 = b + 1 - 1 = b$. \square

Integers \mathbb{Z}

List: 0 (-1) 1 -2 2 -3 (3) -4 4 -5 5...

0 1 2 3 4 5 6 7 8 9 10

$f(-1) = -2(-1) - 1 = 2 - 1 = 1$ $f(3) = 2 \cdot 3 = 6$

$$f : \mathbb{Z} \rightarrow \mathbb{N} \text{ with } f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

Claim: f is a bijection. Proof: Ex.

well defined
one to one
onto

Corollary: $|\mathbb{Z}| = |\mathbb{N}|$

$$|\mathbb{N}| = |\mathbb{Z}^+| = |\mathbb{Z}^-| = |\mathbb{Z}|$$

More examples of countably infinite sets

Claim: S is countably infinite

Similarly: The set of all strings over a specific alphabet is countably infinite.

Bijection using alphabetical-ish ordering (first order by length, then alphabetically among strings of same length) of strands

A, C, G, U, AA, AC, AG, AU, CA, CC, ...
 0 1 2 3 4 5 6 7 8 9

Claim: L is countably infinite

$$|\mathbb{N}| \leq |L|$$

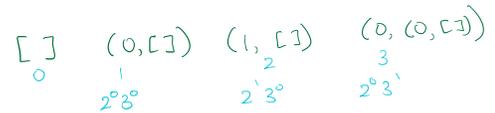
Witness function

$$\begin{aligned} list : \mathbb{N} &\rightarrow L \\ list(n) &= (n, []) \end{aligned}$$



Witness function

$$\begin{aligned} toNum : L &\rightarrow \mathbb{N} \\ toNum([]) &= 0 \end{aligned}$$



$$toNum((n, l)) = 2^n 3^{toNum(l)} \quad \text{for } n \in \mathbb{N}, l \in L$$

1-1? $\forall m \in \mathbb{N} \forall n \in \mathbb{N} (list(m) = list(n) \rightarrow m = n)$?
 Arb m, n . Assume $list(m) = list(n)$. By def $list(m) = (m, [])$, $list(n) = (n, [])$
 so assumption is $(m, []) = (n, [])$, which guarantees $m = n$ \square

$$\begin{aligned} \text{eg } toNum((2, (1, (3, [])))) &= 2^2 3^{toNum((1, (3, [])))} \\ &= 2^2 3^{2^1 3^{toNum(3, [])}} \\ &= 2^2 3^{2^1 3^3} = 2^2 3^{2^1 3^3} = 2^2 3^{2^1 3^3} = 2^2 3^{2^1 3^3} \end{aligned}$$

WTS $toNum$ is one-to-one function
 Namely WTS for any l_1, l_2 $toNum(l_1) = toNum(l_2)$ guarantees $l_1 = l_2$

Claim: $|\mathbb{Z}^+| = |\mathbb{Q}|$

One-to-one function from \mathbb{Z}^+ to \mathbb{Q} is $f_1 : \mathbb{Z}^+ \rightarrow \mathbb{Q}$ with $f_1(n) = n$ for all $n \in \mathbb{N}$.

Remains to show $|\mathbb{Q}| \leq |\mathbb{Z}^+|$. We will do this in three steps



$$f_2 : \mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{Z}$$

$$f_2(x) = \begin{cases} (0, 1) & \text{if } x = 0 \\ (p, q) & \text{if } x = \frac{p}{q}, \\ & \text{gcd}(p, q) = 1, q > 0 \end{cases}$$

"rationals represented uniquely by fractions in lowest terms"

$$f_3 : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$$

$$f_3((x, y)) = \begin{cases} (2x + 2, 2y + 2) & \text{if } x \geq 0, y \geq 0 \\ (-2x - 1, 2y + 2) & \text{if } x < 0, y \geq 0 \\ (2x + 2, -2y + 1) & \text{if } x \geq 0, y < 0 \\ (-2x - 1, -2y - 1) & \text{if } x < 0, y < 0 \end{cases}$$

$$f_4 : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \\ f_4((x, y)) = 2^x 3^y \quad \text{for } x, y \in \mathbb{Z}^+$$

coords packaged into single number



$$\begin{aligned} f_3((0, 0)) &= (2, 2) \\ f_3((1, 1)) &= (3, 3) \\ f_3((-1, -1)) &= (1, 1) \\ f_3((1, -1)) &= (4, 3) \\ f_3((-1, 1)) &= (1, 4) \end{aligned}$$

Notice: $\exists A \exists B (A \neq B \wedge |A| = |B|)$
 Notice: $\forall A \forall B ((|A| = |\mathbb{N}| \wedge |B| = |\mathbb{N}|) \rightarrow |A| = |B|)$
 Cardinality categories

A set A is **finite** means it is empty or it is the same size as $\{1, \dots, n\}$ for some $n \in \mathbb{N}$.

A set A is **countably infinite** means it is the same size as \mathbb{N} . Notice: all countably infinite sets are the same size as each other.

A set A is **countable** means it is either finite or countably infinite.

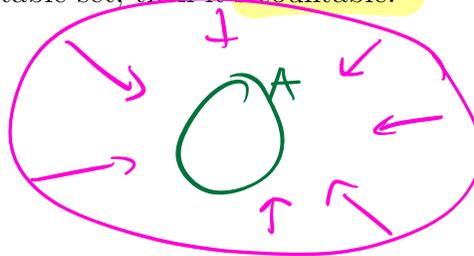
A set A is **uncountable** means it is not countable.

← We don't have an example yet.

Lemmas about countable and uncountable sets

Lemma: If A is a subset of a countable set, then it's countable.

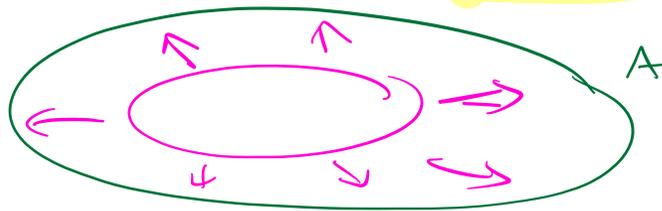
extra practice



- ① Translate to symbols
- ② Identify proof strategies
- ③ Carry out strategies

Lemma: If A is a superset of an uncountable set, then it's uncountable.

extra practice



Lemma: If A and B are countable sets, then $A \cup B$ is countable and $A \cap B$ is countable.

extra practice

Lemma: If A and B are countable sets, then $A \times B$ is countable.

Generalize pairing ideas from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to \mathbb{Z}^+

extra practice

Lemma: If A is a subset of B , to show that $|A| = |B|$, it's enough to give one-to-one function from B to A or an onto function from A to B .

extra practice

Are there any uncountable sets?

Are there always *bigger* sets? **Yes!**

Recall: When U is a set, $\mathcal{P}(U) = \{X \mid X \subseteq U\}$

Key idea: For finite sets, the power set of a set has strictly greater size than the set itself. Does this extend to infinite sets?

Definition: For two sets A, B , we use the notation $|A| < |B|$ to denote $(|A| \leq |B|) \wedge \neg(|A| = |B|)$.

For finite sets

Size			Size of power set	
0	$\emptyset = \{\}$	$\mathcal{P}(\emptyset) = \{\emptyset\}$	1	$ \emptyset < \mathcal{P}(\emptyset) $
1	$\{1\}$	$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$	2	$ \{1\} < \mathcal{P}(\{1\}) $
2	$\{1, 2\}$	$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	4	$ \{1, 2\} < \mathcal{P}(\{1, 2\}) $

N and its power set

Example elements of N

1 5 0

Example elements of $\mathcal{P}(N)$

$\{20\}$

\emptyset empty set

N

$\{x \in N \mid x \bmod 2 = 0\}$

Claim: $|N| \leq |\mathcal{P}(N)|$

Pf: WTS $\exists f: N \rightarrow \mathcal{P}(N)$ (f is one-to-one)

Consider $f: N \rightarrow \mathcal{P}(N)$ given by $f(n) = \{n\}$

for each $n \in N$

other options for witness also work...

Domain of quantification? f is well-defined function with domain N and codomain $\mathcal{P}(N)$ ✓

Satisfies the predicate? WTS $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$

By universal generalization, consider arbitrary $a, b \in N$

Assume $f(a) = f(b)$. WTS $a = b$.

By definition of f , assumption can be rewritten as $\{a\} = \{b\}$, which by definition of set equality guarantees $a = b$ ■

Notice $f(n) = \emptyset$ is not one-to-one so not a witness.

Claim: There is an uncountable set. Example: $\mathcal{P}(\mathbb{N})$

uncountable means infinite and not same size as \mathbb{N} .

Proof: By definition of countable, since $\mathcal{P}(\mathbb{N})$ is not finite, to show is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, to show is

$$\neg \exists f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}) \quad (f \text{ is bijection})$$

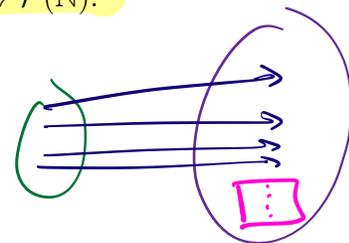
WTS $\forall f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}) \quad (f \text{ is not a bijection})$

Towards a proof by universal generalization, consider an arbitrary function $f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, to show:

$$\neg \forall B \in \mathcal{P}(\mathbb{N}) \exists a \in \mathbb{N} (f(a) = B)$$



. By logical equivalence, we can write this as an existential statement:

$$\exists B \in \mathcal{P}(\mathbb{N}) \forall a \in \mathbb{N} (f(a) \neq B)$$

In search of a witness, define the following collection of nonnegative integers:

Given specific f : $D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$ *Diagonally Disagreeable*

. By definition of power set, since all elements of D_f are in \mathbb{N} , $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N} (f(a) \neq D_f)$.

Proof of lemma: Consider arbitrary integer a . WTS $f(a) \neq D_f$.
 Towards proof by cases, notice that $a \in f(a)$ or $a \notin f(a)$.
 Case ① Assume $a \in f(a)$. By def of D_f , $a \notin D_f$. So: $a \in f(a) \oplus a \notin D_f$
 Case ② Assume $a \notin f(a)$. By def of D_f , $a \in D_f$. So: $a \notin f(a) \oplus a \in D_f$
 Hence, a is witness to $\exists x \in \mathbb{N} (x \in f(a) \oplus x \in D_f)$, i.e. $\neg \forall x \in \mathbb{N} (x \in f(a) \rightarrow x \in D_f)$
 i.e. $f(a) \neq D_f$ \square

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from \mathbb{N} to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would “disagree” with each of the images of f about some element.

subsets of \mathbb{N} defined by choices of $x \in X_n$?

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_f$?
0	$f(0) = X_0$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
1	$f(1) = X_1$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
2	$f(2) = X_2$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
3	$f(3) = X_3$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
4	$f(4) = X_4$	Y / N	Y / N	Y / N	Y / N	Y / N	...	N / Y
⋮								

Countable vs. uncountable: sets of numbers

Comparing \mathbb{Q} and \mathbb{R}

Both \mathbb{Q} and \mathbb{R} have no greatest element.

Both \mathbb{Q} and \mathbb{R} have no least element.

The quantified statement

$$\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$$

is true about both \mathbb{Q} and \mathbb{R} .

Both \mathbb{Q} and \mathbb{R} are infinite. But, \mathbb{Q} is countably infinite whereas \mathbb{R} is uncountable.

The set of real numbers

$$\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

Order axioms (Rosen Appendix 1):

Reflexivity	$\forall a \in \mathbb{R} (a \leq a)$
Antisymmetry	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} ((a \leq b \wedge b \leq a) \rightarrow (a = b))$
Transitivity	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} \forall c \in \mathbb{R} ((a \leq b \wedge b \leq c) \rightarrow (a \leq c))$
Trichotomy	$\forall a \in \mathbb{R} \forall b \in \mathbb{R} ((a = b \vee b > a \vee a < b))$

Completeness axioms (Rosen Appendix 1):

Least upper bound	Every nonempty set of real numbers that is bounded above has a least upper bound
Nested intervals	For each sequence of intervals $[a_n, b_n]$ where, for each n , $a_n < a_{n+1} < b_{n+1} < b_n$, there is at least one real number x such that, for all n , $a_n \leq x \leq b_n$.

Each real number $r \in \mathbb{R}$ is described by a function to give better and better approximations

$$x_r : \mathbb{Z}^+ \rightarrow \{0, 1\} \quad \text{where } x_r(n) = n^{\text{th}} \text{ bit in binary expansion of } r$$

r	Binary expansion	x_r
0.1	0.00011001...	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n > 1 \text{ and } (n \bmod 4) = 2 \\ 0 & \text{if } n = 1 \text{ or if } n > 1 \text{ and } (n \bmod 4) = 3 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 0 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 1 \end{cases}$
$\sqrt{2} - 1 = 0.4142135\dots$	0.01101010...	Use linear approximations (tangent lines from calculus) to get algorithm for bounding error of successive operations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximation that has error less than $2^{-(n+1)}$.

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is uncountable.

Approach 1: Mimic proof that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not finite, **to show** is $|\mathbb{N}| \neq |\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}|$.

To show is $\forall f : \mathbb{Z}^+ \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ (f is not a bijection). Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{Z}^+ \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$. **To show:** f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show:**

$$\exists x \in \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\} \forall a \in \mathbb{N} (f(a) \neq x)$$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1b_2b_3 \dots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of $f(j)$. Since $d_f \neq f(a)$ for any positive integer a , f is not onto.

Approach 2: Nested closed interval property

To show $f : \mathbb{N} \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not onto. **Strategy:** Build a sequence of nested closed intervals that each avoid some $f(n)$. Then the real number that is in all of the intervals can't be $f(n)$ for any n . Hence, f is not onto.

Consider the function $f : \mathbb{N} \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

n	$f(n)$	Interval that avoids $f(n)$
0	0.5	
1	0.920735...	
2	0.954649...	
3	0.570560...	
4	0.121599...	
\vdots		

Other examples of uncountable sets

- The power set of any countably infinite set is uncountable. For example:

$$\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{Z}^+), \mathcal{P}(\mathbb{Z})$$

are each uncountable.

- The closed interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, any other nonempty closed interval of real numbers whose endpoints are unequal, as well as the related intervals that exclude one or both of the endpoints.
- The set of all real numbers \mathbb{R} is uncountable and the set of irrational real numbers $\overline{\mathbb{Q}}$ is uncountable.

¹There's a subtle imprecision in this part of the proof as presented, but it can be fixed.